Evaluating Energy Distribution in Torsional Modes

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Integrated Systems Research, Inc.

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steve.carmichael@isrtechnical.com

Abstract:

This Tech Brief discusses the basis and process of computing the relative kinetic and potential energy in free torsional vibration patterns. The utility of these computations is that they facilitate identifying where, in a system, stiffness and inertia changes will provide the greatest leverage for influencing the system's torsional natural frequencies and inertia participation.

Background:

Typically, torsional natural frequency evaluations are performed for drive train systems having shafts operating at different speeds. The drive elements are oftentimes modeled with a lumped inertia/torsional spring representation and Figure 1 provides a schematic of a simple generic drive and torsional model.

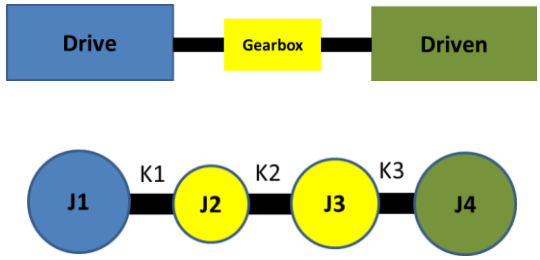

Figure 1

Any shaft in the system can be chosen as the reference to which the inertia and stiffness is reflected, but characteristically it is the driven shaft. The values are reflected to the reference shaft so as not to violate the conservation of energy in the system model. If the gearbox is a reducer and the reference is the drive shaft then the high speed components will be reflected by the square of the gearbox reduction.

In the simple example the gearbox is a reducer with a 10:1 ratio. The inertia and stiffness values for the system are provided in Table 1.

Table 1 – Simple Torsional Model

Inertia and Stiffness Values - Gearbox Reduction 10:1							
Values at Operating Shafts				Values Reflected to Driven Shaft			
	in-lbs-sec^2		In-Lbs/Rad		in-lbs-sec^2		In-Lbs/Rad
J1	55	K1	5.00E+06	J1	5500	K1	5.00E+08
J2	15	K2	2.50E+07	J2	1500	K2	2.50E+09
J3	2000	K3	5.00E+09	J3	2000	K3	5.00E+09
J4	3500			J4	3500		

Eigenvectors or Mode Shapes:

The eigenvalue solution to a spring-mass system provides the frequencies at which a free exchange between kinetic and potential energy occurs in the system. Typically these frequencies should be avoided during operation since they result in amplified vibration responses and would result in an unbounded response without damping being present. Additionally, the eigenvalue solution provides the relative vibration patterns, or eigenvectors, for these states of free energy exchanges.

Frequently, the relative vibration patterns or eigenvectors are normalized to the mass matrix. If this is the case the following relationship holds:

$$\{\phi\}^T [M]\{\phi\} = 1.0$$

Where ϕ is the eigenvector

Equation 1.0

To determine whether or not the extracted vectors have been normalized to the mass matrix, simply look at the value of the vector for the rigid body (e.g., 0.0 Hz) state. The vector will be of uniform value, and if that value is equal to the inverse of the square root of the sum of the system's inertia then the mode shapes (e.g., eigenvectors) have been normalized to the mass matrix.

The natural frequencies and mode shapes for the system defined in Table 1 are provided below:

Table 2 – Eigenvalues and Vectors

	Mode Shape			
Mode	**1**	**2**	**3**	**4**
Frequency (Hz)	0.00	59.48	215.09	366.52
Station				
1	8.94E-03	1.00E-02	-1.13E-03	1.65E-04
2	8.94E-03	-5.38E-03	2.16E-02	-9.45E-03
3	8.94E-03	-8.01E-03	2.48E-03	1.87E-02
4	8.94E-03	-8.87E-03	-8.91E-03	-6.89E-03

The total sum of the system's inertia is 12,500 in-lbs. sec^2. The inverse of the square root of this value is 0.00894. This is the same value of the mode shape for the rigid body mode providing verification that the mode shapes are normalized to the mass matrix.

Relative Potential Energy per Mode:

Evaluating the relative potential energy in each mode enables the engineer to quickly determine which element in the system will have the greatest impact on the frequency of a given mode if its stiffness is changed.

Since the mode shapes are normalized to the mass matrix there is no need to obtain an absolute displacement field. This would have been done by scaling the vectors with a mode coefficient based on uniform torsional excitation. The vector can be treated as the actual torsional angular displacement of the drive, for a given mode, since the evaluation is relative not absolute. The units for the vector are used as if they are radians (e.g., as if the vector had been scaled by a mode coefficient).

The relative potential energy per span per mode is computed by:

$$PE_n = 0.50K_j\left(\phi_j - \phi_k\right)^2$$

Equation 2.0

Where the subscript *n* is the specific span (1-3), *j* is the station and *k* is *j*+1.

For the simple system under consideration the relative potential energy is provided in Table 3:

Table 3 – Relative Potential Energy

Span	Relative Potential Energy Per Mode			
	1	2	3	4
1	0.00E+00	5.93E+04	1.30E+05	2.31E+04
2	0.00E+00	8.64E+03	4.59E+05	9.91E+05
3	0.00E+00	1.88E+03	3.25E+05	1.64E+06
Total	0.00E+00	6.98E+04	9.13E+05	2.65E+06
Span	**Percent Potential Energy Per Mode**			
	1	**2**	**3**	**4**
1	NA	84.9%	14.2%	0.9%
2	NA	12.4%	50.3%	37.4%
3	NA	2.7%	35.5%	61.8%
Total		100.0%	100.0%	100.0%

Relative Kinetic Energy per Mode:

Computing the relative kinetic energy enables efficient decisions regarding the impact changes in inertia will have on a given mode. The relative kinetic energy per lumped inertia is given in Table 4.

An important cross check is to ensure that the kinetic energy is equal to the potential energy for each mode. This will be the case since the mode is at a frequency where there is a free exchange between the two forms of energy.

Table 4 – Relative Kinetic Energy

Station	Relative Kinetic Energy Per Mode			
	1	2	3	4
1	0.00E+00	3.86E+04	6.46E+03	3.96E+02
2	0.00E+00	3.03E+03	6.42E+05	3.55E+05
3	0.00E+00	8.95E+03	1.12E+04	1.85E+06
4	0.00E+00	1.92E+04	2.54E+05	4.41E+05
Total	0.00E+00	6.98E+04	9.13E+05	2.65E+06
Span	**Percent Kinetic Energy Per Mode**			
	1	**2**	**3**	**4**
1	NA	55.3%	0.7%	0.0%
2	NA	4.3%	70.3%	13.4%
3	NA	12.8%	1.2%	69.9%
4	NA	27.6%	27.8%	16.6%
Total		100.0%	100.0%	100.0%

The relative kinetic energy is calculated by equation 3.0. Where j is the station for a given mode

$$KE_j = 0.50J_j\dot{\theta}_j^{\,2}$$

Equation 3.0

Assuming simple harmonic behavior, the motion of the lumped inertia is given in equation 4.0. The mode shape, as in equation 2.0, is treated as relative displacements (radians) in the vibration pattern associated with the natural frequency Ω.

$$\{\theta\} = \{\phi\}\cos(\Omega t)$$

Equation 4.0

$$\{\dot{\theta}\} = -\{\phi\}\Omega\sin(\Omega t)$$

Equation 5.0

Substituting equation 5.0 into equation 3.0:

$$KE_j = 0.50J_j\left(\phi_j\Omega\right)^2$$

Given the distribution of the kinetic energy and stiffness of this simple system, the first elastic mode is primarily controlled by the inertia of the driver. The second and third elastic modes are governed by the inertia of the gearbox.

Summary:

Treating the eigenvectors as relative angular displacements in a torsional vibration pattern enables an engineer to quickly determine the potential and kinetic energy distribution in a given mode. This provides the key parameters for determining the best approach for moving the frequencies of modes or decoupling them.

The key to this approach is to ensure the mode shapes are normalized to the mass matrix. Checking to see that the total inertia of the system has been accounted for in the rigid body mode is a means by which an analyst can confirm whether or not the eigenvectors have been normalized to the mass matrix.

Additionally, the other cross check that should be done is to ensure that the potential and kinetic energies are equal for each mode. This ensures that the computations for the relative energies have been correctly performed.